A two-step approach to aircraft conflict resolution combining optimal deterministic design with Monte Carlo stochastic optimization

Maria Prandini, Luigi Piroddi, and John Lygeros

Abstract— **This paper proposes a novel approach to aircraft conflict resolution where the design of an optimal conflict resolution manoeuvre based on the aircraft intent information is robustified against the uncertainty affecting the aircraft future positions by a randomized stochastic optimization method. The goal is to account for a probabilistic description of the uncertainty affecting the aircraft motion, while avoiding the excessive computational load of a pure Monte Carlo stochastic optimization method.**

I. Introduction

In the airspace over Europe, air traffic control of en-route flights is managed through 75 Air Traffic Control Centers (ATCCs) partitioned into sectors, each controlled by a team of two or three Air Traffic Controllers (ATCs). Sectors are designed so that the nominal flow of traffic through each sector can be safely handled by the ATCs in charge of that sector. The ATCC capacity is limited by the sector with the minimum capacity, and, hence, the capacity of the overall Air Traffic Management (ATM) system is limited by the workload levels that ATCs can sustain.

The growth in air traffic demand is pushing the ATM system to its limit. As reported in [1], the average daily traffic above Europe in 2006 grew by 4.1% with respect to the preceding year, with a 4.6% increase of the total delay, much higher than expected. Introducing tools to assist ATCs and reduce their workload is therefore becoming crucial for adapting the ATM system capacity to the increased demand, thus reducing delays while, at the same time, guaranteeing safety in air travel.

ATCs are responsible for maintaining the appropriate separation between aircraft and avoiding conflicts. A conflict is the event where two aircraft get closer than a minimum safety distance, which is defined according to specific separation criteria. Aircraft separation criteria are intimately connected with the expected air traffic density and the level of technological advancement of the ATM system. Currently, the en-route separation criterion depends on the phase of the flight and other factors and is specified in terms of a minimum vertical distance (typically 1000 ft) and minimum horizontal distance between aircraft at the same flight level (3 nmi near airports and 5 nmi otherwise).

Several critical situations can occur in the management of aircraft separation, during all phases of the flight. Various methods for automatic Conflict Detection and Resolution (CDR) have been proposed in the literature (see [2] for a comprehensive review) with the goal of enhancing the aircraft separation management function in several conditions. An essential element of a CDR method is the model adopted for predicting the aircraft future positions during the look-ahead time horizon. If a loss of separation is detected based on the prediction model, an alarm is issued and a conflict resolution manoeuvre is possibly suggested.

When the prediction time horizon is short (a few seconds to 1 minute), the nominal aircraft trajectory can be used. Over larger time horizons (the so-called mid-term CDR) prediction of future aircraft positions has to account for uncertainty, due to measurement imprecision and noise, delayed information updating, and errors in tracking the prescribed flight plan. To this purpose, the worst-case approach to CDR detects a conflict if there exists at least a pair of trajectories that violates the minimum separation in the set of all admissible aircraft trajectories. The probabilistic approach represents a good trade-off between the nominal and worst-case approaches, since the likelihood of the different admissible trajectories is considered when assessing the possibility that a conflict occurs.

In this paper, conflict resolution is studied on a mid-term time horizon of 20-30 minutes. The model adopted for predicting the aircraft future positions is probabilistic. The objective is to determine a resolution manoeuvre that maximizes a given performance criterion related, *e.g.*, to passenger comfort, fuel consumption, etc., while guaranteeing a low probability of conflict. This naturally leads to a constrained stochastic optimization problem, difficult to solve analytically. Based on the reformulation of the constrained optimization problem as an unconstrained optimization problem with a penalized performance function, a randomized algorithm is proposed in [3] to determine a sub-optimal solution. In that algorithm candidate resolution manoeuvres are extracted at random according to a probability distribution that is proportional to the penalized performance index to be maximized, so that they concentrate close to optimal or nearly-optimal resolution manoeuvres. In principle, this randomized solution allows to use arbitrarily complex (and supposedly realistic) prediction models, but, in practice, the resulting computational effort hampers its applicability.

Starting from the analysis of the critical aspects of the Monte Carlo Markov Chain (MCMC) implementation of the randomized solution in [3], we propose here an algorithm with significantly enhances performance. The key idea is to combine the MCMC algorithm with a deterministic method

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that computes the optimal conflict-free resolution manoeuvre by relying on the nominal aircraft trajectories and neglecting the probabilistic component affecting the prediction of the aircraft future positions (see e.g. [4]-[8]). The resulting two-step approach can be viewed as a methodology to robustify the nominal design obtained via deterministic optimal conflict resolution against the uncertainty affecting the future aircraft positions.

The paper is organized as follows. Section II illustrates the Monte Carlo optimization approach to conflict resolution introduced in [3]. Section III proposes the combination of this approach with optimization techniques based on deterministic models for the prediction of future aircraft position. Finally, some concluding remarks are presented in Section IV.

II. Monte Carlo optimization for aircraft conflict resolution

Consider N aircraft flying in a certain airspace region during a given look-ahead time horizon $[0, t_f]$, where 0 represents the current time instant and t_f is of the order of 20-30 minutes.

According to the current centralized ATM system, each aircraft is assigned a flight plan described by a sequence of way-points, *i.e.*, three-dimensional points with associated arrival times. The aircraft should track the reference path given by the sequence of line segments between subsequent way-points, while trying to meet the scheduled arrival times.

Let the stochastic process $\boldsymbol{\omega}_i$ that describes the aircraft future positions during the time horizon $[0, t_f]$ be defined on the probability space $(\Omega, \mathcal{F}, P)$, where the sample space Ω represents the set of all possible N-tuple of trajectories. The characteristics of process $\boldsymbol{\omega}_i$ are determined by the flight plan of aircraft i, the aircraft dynamics, the actions of the flight management system, and by the uncertainty that affects its current state measurement and its motion. Considering two aircraft, i and j, and letting $\boldsymbol{\Delta\omega}_{ij} = \boldsymbol{\omega}_i - \boldsymbol{\omega}_j$, then, the probability of a conflict during $[0, t_f]$ can be expressed as:

$$P_c = P\left(\boldsymbol{\Delta\omega}_{ij}(t) \in C \text{ for some } t \in [0, t_f] \text{ and } (i,j),\ i \neq j\right),$$

where

$$C = \{(c_1, c_2, c_3) \in \mathbb{R}^3 : \sqrt{c_1^2 + c_2^2} \leq r, |c_3| \leq h\} \qquad (1)$$

is a cylinder of radius r and height $2h$ centered at the origin representing the conflict zone, r and h being the minimum horizontal and vertical distances for aircraft separation.

The strategy adopted by the ATCs when a conflict is likely to occur consists of redefining the flight plan of the aircraft involved in the predicted conflict, either adding new way-points or modifying the coordinates of the existing ones, so as to minimize the probability of conflict occurrence, while satisfying performance requirements (such as fuel saving, or passenger comfort, etc.).

As explained next, this strategy can be translated in mathematical terms as a constrained stochastic optimization problem.

Let u taking values in some compact set $\mathcal{U} \subset \Re^p$ denote the parameterization of the joint resolution manoeuvre involving the N aircraft. u can represent, for instance, the coordinates of additional intermediate way-points of the N aircraft flight plans. Different values for $u \in \mathcal{U}$ result in different likelihoods of the N aircraft trajectories. Thus, the probability space $(\Omega, \mathcal{F}, P_u)$ describing the future motion of the N aircraft has a probability measure P_u that depends on $u \in \mathcal{U}$.

If we denote by $\Omega_A \subseteq \Omega$ the set of conflict-free N-tuples of trajectories, the probability that a conflict occurs during $[0, t_f]$ is a function of $u \in \mathcal{U}$ which can be expressed as

$$P_c(u) = \int_{\Omega \setminus \Omega_A} P_u(d\omega). \qquad (2)$$

Let $\textit{perf}(u, \omega) : \mathcal{U} \times \Omega \to [0, 1]$ be a function that associates to a joint resolution manoeuvre with parameter $u \in \mathcal{U}$ and an N-tuple of trajectories $\omega \in \Omega_A$ a performance index related, *e.g.*, to fuel consumption and passenger comfort. If $\omega \notin \Omega_A$, then $\textit{perf}((u, \omega) = 0$ for any $u \in \mathcal{U}$ since the N-tuple ω of trajectories is not conflict-free. The expected performance of the joint resolution manoeuvre with parameter $u \in \mathcal{U}$ is then given by

$$\textit{PERF}(u) = E_u[\textit{perf}((u, \boldsymbol{\omega})] = \int_\Omega \textit{perf}((u, \omega) P_u(d\omega).$$

The problem of choosing an appropriate joint resolution manoeuvre can then be reformulated as that of choosing $u \in \mathcal{U}$ so as to maximize the expected performance, while guaranteeing that the probability of conflict does not exceed a given threshold $\bar{P}$:

$$\sup_{u \in \mathcal{U}} \textit{PERF}(u) \text{ subject to } P_c(u) \leq \bar{P}. \qquad (3)$$

This constrained optimization problem can be reformulated as an unconstrained one by introducing a suitable term in the performance index so as to favor those values of $u \in \mathcal{U}$ that make conflict-free N-tuples of trajectories more likely to occur. More precisely, let

$$v(u, \omega) = \begin{cases} \textit{perf}(u, \omega) + \Lambda, & \omega \in \Omega_A \\ 1, & \omega \notin \Omega_A \end{cases} \qquad (4)$$

where Λ is a reward for the satisfaction of the constraint. Λ is always greater than 1, so that if $\omega \in \Omega_A$, then $v(u, \omega) > 1$, $\forall u \in \mathcal{U}$. For a given $u \in \mathcal{U}$, let

$$V(u) = E_u[v(u, \boldsymbol{\omega})] = \int_\Omega v(u, \omega) dP_u(\omega). \qquad (5)$$

be the expected value of $v(u, \boldsymbol{\omega})$ with $\boldsymbol{\omega}$ distributed according to P_u.

The following proposition characterizes the performance achievable by solving the unconstrained problem

$$\sup_{u \in \mathcal{U}} V(u) \qquad (6)$$

in place of the original constrained problem (3).

Proposition 1 ([3]): Assume that the supremum in (6) is attained and let $u^\star = \arg\sup_{u\in\mathcal{U}} V(u)$. Then,

$$
\begin{aligned}
&P_c(u^\star) \le \frac{1}{\Lambda} + \big(1-\frac{1}{\Lambda}\big)P_{\min}\\
&\textit{PERF}(u^\star) \ge \textit{PERF}_{\max} - (\Lambda-1)(\bar{P}-P_{\min}) \qquad (7)
\end{aligned}
$$

where $P_{\min} = \inf_{u\in\mathcal{U}} P_c(u)$ and $\textit{PERF}_{\max} = \sup_{u\in\{u\in\mathcal{U}:\, P_c(u)\le\bar{P}\}} \textit{PERF}(u)$. □

Proposition 1 can be used to select Λ so as to ensure that $u^\star$ satisfies $P_c(u^\star) \le \bar{P}$. In fact, if $\tilde{u} \in \mathcal{U}$ is such that $P_c(\tilde{u}) \le \bar{P}$, then Λ has to satisfy $\Lambda \ge \frac{1-P_c(\tilde{u})}{\bar{P}-P_c(\tilde{u})}$ in order to obtain $P_c(u^\star) \le \bar{P}$. If, in addition, $P_{\min} = 0$, then $\Lambda \ge \frac{1}{\bar{P}}$ ensures that $P_c(u^\star) \le \bar{P}$. Since $\textit{PERF}(u^\star)$ worsens with increasing Λ (see equation (7)), the selection of Λ that reduces the solution sub-optimality is $\Lambda = \frac{1-P_c(\tilde{u})}{\bar{P}-P_c(\tilde{u})}$ and $\Lambda = \frac{1}{\bar{P}}$, respectively.

The problem of determining $u^\star = \arg\sup_{u\in\mathcal{U}} V(u)$ is not easy to solve, since it involves determining the expected value of the performance $v(u,\boldsymbol{\omega})$ and minimizing it with respect to u. In general, the expected value of $v(u,\boldsymbol{\omega})$ cannot be computed analytically given that P_u is defined indirectly as the probability measure on the N-tuples of trajectories induced by the prediction model of the aircraft future positions. Also, even if an analytic expression for $V(u)$ were available, it would be generally difficult to determine its maximum.

The proof of Proposition 1 in [3] requires that function $\textit{perf}(u,\omega)$ satisfies $0 \le \textit{perf}(u,\omega) \le 1$, $\forall (u,\omega) \in \mathcal{U}\times\Omega_A$. Such condition can be disadvantageous with respect to the optimization of $V(u)$, since $\Lambda > 1$ is added to $\textit{perf}(u,\omega)$ in the definition of $v(u,\omega)$ (see (4)). A large $\Lambda > 1$ yields a nearly flat function $v(u,\omega)$ over Ω_A, which makes it difficult to identify the parameter $u^\star$ with the best performance among those $u \in \mathcal{U}$ that cause most of the N-tuples to be conflict-free. This undesired effect of a large Λ can be counteracted by replacing the performance function $V(\cdot)$ with $V^M(\cdot)$, where M is an integer greater than 1: the larger is M, the more peaked is $V^M(\cdot)$.

A. MCMC solution to the unconstrained optimization

Inspired by the randomized approach to stochastic optimization proposed in [9], [10], [11], the problem of maximizing $V^M(\cdot)$ is addressed in [3] by introducing a fictitious random variable $\mathbf{u}$ taking values in $\mathcal{U} \subset \Re^p$ and with probability density $f_t(\cdot)$ proportional to $V^M(\cdot)$ ($f_t(\cdot) \propto V^M(\cdot)$). In this way, the extractions of $\mathbf{u}$ will concentrate close to the mode of $f_t(\cdot)$, and, hence, to the maximum of $V^M(\cdot)$.

Let us introduce the set of random variables $(\mathbf{u},\boldsymbol{\omega}^{(1)},\boldsymbol{\omega}^{(2)},\dots,\boldsymbol{\omega}^{(M)})$ on the extended probability space $(\Omega_e,\mathcal{F}_e,P_e)$ where P_e is such that $P_e\big(\mathbf{u},\boldsymbol{\omega}^{(1)},\boldsymbol{\omega}^{(2)},\dots,\boldsymbol{\omega}^{(M)}) \in A_u \times A_\omega\big) \propto \int_{A_u}\big(\int_{A_\omega}\prod_{i=1}^M v(u,\omega^{(i)})P_u(d\omega^{(i)})\big)du$. Then, $\mathbf{u}$ has a probability density

$$
f_t(\cdot) \propto \int_{\Omega^M}\prod_{i=1}^M v(\cdot,\omega^{(i)})P_u(d\omega^{(i)}) = V^M(\cdot).
$$

The Metropolis-Hastings (MH) algorithm, [12], [13], can be used to extract values for the introduced set of random variables $(\mathbf{u},\boldsymbol{\omega}^{(1)},\boldsymbol{\omega}^{(2)},\dots,\boldsymbol{\omega}^{(M)})$. It is in fact easier to extract joint values for $(\mathbf{u},\boldsymbol{\omega}^{(1)},\boldsymbol{\omega}^{(2)},\dots,\boldsymbol{\omega}^{(M)})$ than directly for $\mathbf{u}$ only, since this would involve computing the expectation in (5).

The MH algorithm belongs to the so-called Monte Carlo Markov Chain (MCMC) algorithms, whose objective is the extraction of samples from a "target distribution", constructed as the limit distribution of an ergodic Markov chain. The implementation of the MH algorithm for aircraft conflict resolution requires a "instrumental density" $f_p(\cdot)$ for extracting values from $\mathcal{U}$ and a prediction model for generating N aircraft trajectories for a given value u of the joint resolution manoeuvre parameter. In principle, it is applicable to any stochastic model for aircraft trajectory prediction, provided that it is possible to simulate it.

MCMC Algorithm for conflict resolution ([3])

Initialization:

extract u_0 *according to* $f_p(\cdot)$

generate M N*-tuples of trajectories* $\omega_0^{(1)}$, $\omega_0^{(2)}$, $\dots,\omega_0^{(M)}$ *when the resolution manoeuvre parameter is* u_0

compute the performance index $v_0 = \prod_{i=1}^M v(u_0,\omega_0^{(i)})$

k-th iteration:

extract $\tilde{u}$ *according to* $f_p(\cdot)$

generate M N*-tuples of trajectories* $\tilde{\omega}^{(1)}$, $\tilde{\omega}^{(2)}$, $\dots,\tilde{\omega}^{(M)}$ *when the resolution manoeuvre parameter is* $\tilde{u}$

compute the performance index $\tilde{v} = \prod_{i=1}^M v(\tilde{u},\tilde{\omega}^{(i)})$

define $a_k = \min\big\{\frac{\tilde{v}}{f_p(\tilde{u})}\frac{f_p(u_{k-1})}{v_{k-1}},1\big\}$

set $(u_k,v_k) = \begin{cases}(\tilde{u},\tilde{v}) & \textit{with probability } a_k\\ (u_{k-1},v_{k-1}) & \textit{with probability } 1-a_k\end{cases}$

According to this algorithm, one selects u_k by extracting a value $\tilde{u}$ from $\mathcal{U}$ according to the instrumental density $f_p(\cdot)$. The performance of the extracted $\tilde{u}$ is tested on M N-tuple of aircraft trajectories: the better is the performance $\tilde{v}$ of $\tilde{u}$ with respect to that of the previously selected u_{k-1}, the more likely is that $\tilde{u}$ would represent an extraction from the Markov chain with limiting density for $\mathbf{u}$ proportional to $V(\cdot)$. Since a non-uniform $f_p(\cdot)$ favors the extraction of certain $\tilde{u}$ with respect to others, the relative likelihood of $\tilde{u}$ and u_{k-1} is considered when deciding if accepting $\tilde{u}$ or setting $u_k = u_{k-1}$.

Under the assumption that the support of the instrumental density $f_p(\cdot)$ contains that of the target density $f_t(\cdot)$, the state u_k generated by the algorithm is distributed according to the desired density $f_t(\cdot)$, in the limit. In practice, one has to define a burn-in period required for the Markov chain to reach the limit behavior. The samples u_k obtained after the burn-in period will be extracted approximately from $f_t(\cdot)$. Some criteria to evaluate when convergence is reached have been proposed in the literature (see e.g. [13]), but this is still subject of ongoing research.

B. Analysis of the critical aspects of the MCMC solution

Even when using a low value for M, the algorithm is capable of clearly separating $\mathcal{U}$ into a region of conflict-free flight plans and a region of high conflict probability. However, in order to discern the regions associated to high performance, a high M is necessary, to force $f_t(\cdot)$ to concentrate around the global optimum (see [14], [15] for a quantitative discussion). Unfortunately, the computational load of the algorithm also increases with M. The obvious reason for this is that M simulations need to be performed at each iteration of the algorithm. Somewhat more subtly, the rate of convergence of the Markov chain to the limit behavior (and hence the length of the burn-in period) also depends on M.

To demonstrate this, let π_k denote the probability distribution of the u_k generated by the algorithm and π_t the target probability distribution (corresponding to the density $f_t(\cdot)$). Standard results in the theory of Markov chains allow one to quantify the rate of convergence of π_k to π_t, under some conditions. The following was shown in [16].

Theorem 1: Assume that there exists $B > 1$ such that for all $u \in \mathcal{U}$, $f_t(u) > 0$, $f_p(u) > 0$, and $f_t(u) \leq B f_p(u)$. Then

$$\|\pi_t - \pi_k\|_{TV} \leq \left(1 - \frac{1}{B}\right)^k.$$

Here $\|\cdot\|_{TV}$ denotes the total variation norm. We see that the distribution of u_k converges to the target distribution geometrically. If we select $\sigma \in (0,1)$ we can then define the burn-in period as the number of steps required to get within σ of the target distribution in the total variation norm. Taking logarithms we see that in this case it suffices to select a burn-in period

$$k \geq \frac{\log \frac{1}{\sigma}}{\log \frac{B}{B-1}}.$$

The conditions of Theorem 1 are easy to satisfy in our case. If, for example, the space $\mathcal{U}$ is compact we can simply take $f_p(\cdot)$ to be the uniform distribution with density

$$f_p(u) = \frac{1}{\lambda(\mathcal{U})} > 0$$

where $\lambda(\mathcal{U})$ denotes the Lebesgue measure of the set $\mathcal{U}$. Moreover, notice that, by construction

$$V(u) = \int_\Omega v(u,\omega) dP_u(\omega) \geq \int_\Omega dP_u(\omega) = 1$$

and

$$f_t(u) = \frac{V(u)^M}{\int_\Omega V(u)^M \lambda(du)} \leq \frac{(1+\Lambda)^M}{\lambda(\mathcal{U})}$$

leading to $B = (1+\Lambda)^M$. Noting that

$$\log\left(\frac{(1+\Lambda)^M}{(1+\Lambda)^M - 1}\right) \geq \frac{1}{(1+\Lambda)^M}$$

we see that it suffices to take a burn-in period

$$k \geq (1+\Lambda)^M \log \frac{1}{\sigma}.$$

Note that the length of the burn-in period is exponential in M. Therefore, if we would like to improve the accuracy of the solution by increasing M we would not only suffer a (linear) increase in the number of extractions needed by each step of the algorithm, but also have to wait (exponentially) longer.

To counteract this effect a better instrumental distribution is needed. Several alternatives present themselves.

1) Make the new extractions depend on the current state. It turns out that in this case it is much harder to bound the rate of convergence [16]. We will not pursued this alternative further here.
2) Develop an annealing schedule for M. Roughly speaking, one can fix a low value of M and run the chain longer than the burn-in period. One can then use the tail of the u_k sequence to generate a new instrumental distribution, e.g. a mixture of Gaussian, possibly after clustering the samples. M can then be increased and the process repeated. Such an approach was explored numerically in [3].
3) Bias the instrumental distribution toward regions of $\mathcal{U}$ that are known to contain good solutions. Such regions can be estimated by solving a deterministic optimization problem first, then biasing the randomized search toward its solutions. This will be the approach pursued in the present paper.

The hope with the third approach is that the deterministic optima will be in the vicinity of the stochastic optima u^*. Therefore, biasing $f_p(\cdot)$ toward the deterministic optima is likely to make $f_p(u^*)$ larger, decreasing the value of B needed to ensure that $f_t(u^*) \leq B f_p(u^*)$ (recall that $f_t(\cdot)$ reaches its maximum at u^*) and speeding up convergence. The danger of course is that if the deterministic optimum happens to be far from the stochastic one, then biasing $f_p(\cdot)$ in this way would imply a larger value of B making convergence slow, possibly slower than with the uniform instrumental distribution.

III. A TWO-STEP SOLUTION TO OPTIMAL CONFLICT RESOLUTION

The idea proposed in this paper is as simple as follows: to increase the convergence speed to the optimal solution, start the algorithm with an instrumental probability density as close as possible to the target one. The instrumental density is built based on the manoeuvre computed by a deterministic method for conflict resolution that neglects the uncertainty affecting the prediction of the aircraft future positions and relies on the nominal aircraft trajectories. In this way, the value for M used in the MCMC algorithm implementation does not need to be much high since the samples extracted from the instrumental density should be already close to the maximum of $V(u)$. The application of the MCMC algorithm to the instrumental density obtained by the deterministic resolution method can be reinterpreted as a way to robustify the nominal design against the uncertainty affecting the aircraft future positions.

We next illustrate this idea with reference to a specific conflict resolution problem, and verify its efficacy on a simple numerical example.

A deterministic solution to the conflict resolution problem which optimizes fuel consumption and passenger comfort based on the nominal aircraft trajectories is presented first. The nominal design is then robustified against uncertainty on the future aircraft behavior by MCMC optimization.

Suppose that aircraft i is following a flight plan that consists of a starting position $a_i \in \mathbb{R}^3$ at time 0 and a destination $b_i \in \mathbb{R}^3$ at time t_f. With reference to the time horizon $[0, t_f]$, a trajectory for the ith aircraft can be expressed as a continuous and piecewise differentiable function $\omega_i : [0, t_f] \to \mathbb{R}^3$, such that $\omega_i(0) = a_i$ and $\omega_i(t_f) = b_i$. The "energy" of ω_i is defined as:

$$J(\omega_i) = \frac{1}{2} \int_0^{t_f} \|\dot{\omega}_i(t)\|^2 \, dt, \tag{8}$$

where $\|\dot{\omega}_i\|$ represents the aircraft velocity.

Trajectory ω_i can be interpreted as a curve in $\mathbb{R}^3$ parameterized in $t \in [0, t_f]$, whose length is given by:

$$L(\omega_i) = \int_0^{t_f} \|\dot{\omega}_i(t)\| \, dt.$$

By the Cauchy-Schwarz inequality, we have that $J(\omega_i) \geq \frac{1}{2} \frac{L(\omega_i)^2}{t_f}$. The equality holds if and only if the velocity $\|\dot{\omega}_i\|$ is constant and, in that case, $J(\omega_i)$ is proportional to $\|\dot{\omega}_i\|^2$. As expected, we can conclude that the energy $J(\omega_i)$ is minimal if and only if trajectory ω_i is the smooth constant-speed motion along the line segment from a_i to b_i, which has a practical impact on fuel consumption and passenger comfort.

Let us now consider N aircraft flying from the initial positions $a = (a_1, a_2, \ldots, a_N)$ to the final positions $b = (b_1, b_2, \ldots, b_N)$ in the time horizon $[0, t_f]$. The joint trajectory of the N aircraft is represented by $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ and is conflict-free only if no aircraft enters the cylindrical protection zone C defined in (1) surrounding any other aircraft during the time horizon $[0, t_f]$. We shall denote the set of conflict-free N-tuples of trajectories starting at a and ending at b as $\Omega_A(a, b)$

Suppose that a conflict would occur if all the N aircraft follow the minimal energy trajectory. To solve this conflict situation, an intermediate way-point is introduced at time $t_c \in (0, t_f)$ with spatial coordinates $u_i \in \mathbb{R}^3$, $i = 1, 2, \ldots, N$, with the understanding that the resulting trajectories are composed of two straight line segments, each one covered at constant speed, that is:

$$\omega_i(t) = \begin{cases} a_i + (u_i - a_i)\frac{t}{t_c}, & 0 \leq t < t_c \\ b_i + (u_i - b_i)\frac{t - t_f}{t_c - t_f}, & t_c \leq t \leq t_f \end{cases} \tag{9}$$

The parameter vector $u = (u_1, u_2, \ldots, u_n)$ has to be appropriately chosen so as to avoid the occurrence of a conflict while not deviating too much from the minimal energy solution.

According to the approach proposed in [8], [17] to optimal (deterministic) conflict resolution, we introduce the "μ-energy" function

$$J_\mu(\omega) = \sum_{i=1}^{N} \mu_i J(\omega_i),$$

where μ_i are positive real numbers adding up to one that represent the priorities of the different aircraft. Based on the parameterization of ω_i in (9), $J_\mu(\omega)$ can be written explicitly as a function of u:

$$J_\mu(\omega) = \frac{1}{2} \frac{t_f}{(t_f - t_c)t_c} \sum_{i=1}^{N} \mu_i \|u_i - u_i^c\|^2 + k, \tag{10}$$

where k is a constant, and

$$u_i^c = \frac{(t_f - t_c)a_i + t_c b_i}{t_f}$$

are the optimal way-points in the absence of the conflict avoidance constraint.

Optimal conflict resolution in this deterministic setting is then formulated as the following constrained optimization problem:

$$\min_{u \in \mathcal{U}} J_\mu(\omega) \text{ subject to } \omega \in \Omega_A(a, b). \tag{11}$$

where $J_\mu(\omega)$ is a quadratic (and hence convex) function of u (see equation (10)). The constraint $\omega \in \Omega_A(a, b)$ can also be expressed in terms of u based on (9). The resulting constraints are non-convex but they can be approximated so as to transform (11) into a convex optimization problem, [8]. Let $\bar{u}^\star$ be the (approximate) solution to the constraint optimization problem (11). The designed joint resolution manoeuvre with parameter $\bar{u}^\star$ is robustified against the uncertainty affecting the aircraft future positions by considering a stochastic prediction model and applying the MCMC algorithm with a Gaussian density with mean $\bar{u}^\star$ as instrumental density $f_p(\cdot)$.

A. Numerical example

Consider two aircraft flying at the same altitude during the time horizon $[0, t_f]$ with $t_f = 20$ minutes. The flight plan of aircraft 1 consists of two way-points: $a_1 = (0, 120)$ at time 0 and $b_1 = (240, 120)$ at time t_f, with the two spatial coordinates identifying the position on the horizontal plane at a fixed altitude measured in kilometers. As for the flight plan of aircraft 2, it is given by $a_2 = (120, 0)$ at time 0 and $b_2 = (120, 240)$ at time t_f.

Clearly, if the two aircraft were to follow the assigned flight plans exactly, a collision would occur (see Figure 1). To avoid that the aircraft get closer than 5 nmi, the flight plans have to be modified by introducing intermediate way-points. We assume, for simplicity, that only the flight plan of aircraft 2 is modified by introducing an intermediate way-point $u = (u_x, u_y)$ at time $t_c = t_f/2 = 10$, at the fix altitude. The spatial coordinates of the introduced way-point should be selected so as to minimize the deviation for the straight line

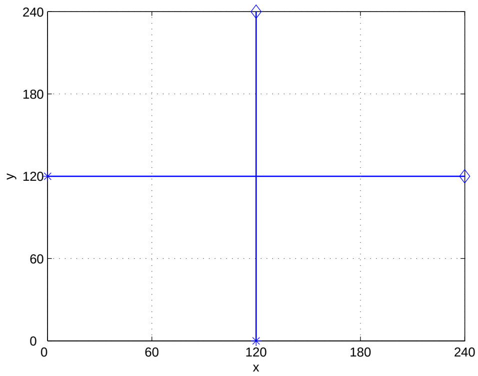

Fig. 1. Flight plan of two aircraft flying at the same altitude. Origin and destination are indicated with $*$ and $\diamond$, respectively.

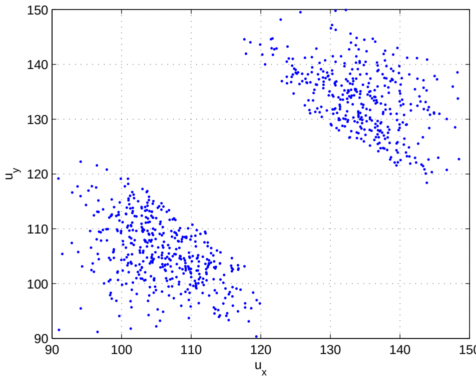

Fig. 2. Values of $u = (u_x, u_y)$ extracted with the 2-steps Monte Carlo optimization method.

path traveled at constant velocity, thus avoiding excessive fuel consumption and discomfort for passengers.

The performance of the resolution manoeuvre with parameter u over the set of conflict-free joint trajectories Ω_A is set equal to

$$perf(u,\omega) = e^{-\frac{\|u-(120,120)\|}{100}}, \ \omega \in \Omega_A$$

to penalize deviations from the straight trajectory as well as velocity modifications.

As a result the penalized performance takes the form

$$v(u,\omega) = \begin{cases} e^{-\frac{\|u-(120,120)\|}{100}} + \Lambda, & \omega \in \Omega_A \\ 1, & \omega \notin \Omega_A. \end{cases}$$

We set the threshold conflict probability $\bar{P} = 0.1$, which corresponds to $\Lambda = 10$ (it is easy to see that effectively conflict free plans exist, hence $P_{\min} \approx 0$).

Note that we need only to be able to to detect if the aircraft get in conflict ($\omega \notin \Omega_A$) or not ($\omega \in \Omega_A$) when simulating the two aircraft trajectories in the MH algorithm implementation. We can then reduce the effort by simulating directly the relative trajectory $\boldsymbol{\Delta\omega} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$ of the two aircraft instead of the trajectories of the two aircraft.

In this numerical example, we adopted a simple stochastic prediction model taken from [18], where the aircraft relative position is governed by the stochastic differential equation

$$d\boldsymbol{\Delta\omega}(t) = \Delta v(t)dt + \rho\sqrt{2[1 - r(\boldsymbol{\Delta\omega})]}d\mathbf{B}(t) \qquad (12)$$

with $\mathbf{B}$ representing a standard 2D Brownian motion. As for the other quantities in (12), $\Delta v(t) = v_1(t) - v_2(t)$ is the aircraft relative velocity, $\rho > 0$ modulates the variance of the standard Brownian motion, and $r(x) = e^{-c\|x\|}$, $c > 0$, is an exponentially decaying spatial correlation term accounting for wind as the main source of uncertainty on the aircraft future position (see [18] for more details). In the simulations we set the spatial correlation coefficient $c = 0.5$ and the variance parameter $\rho = 50$. The values for $u = (u_x, u_y)$ extracted with the Monte Carlo optimization method proposed in [3] are reported in Figure 2. Two runs of the MH algorithm are executed: in the first run we set $M = 10$, whereas in the second run $M = 100$. Clearly, to avoid the conflict,

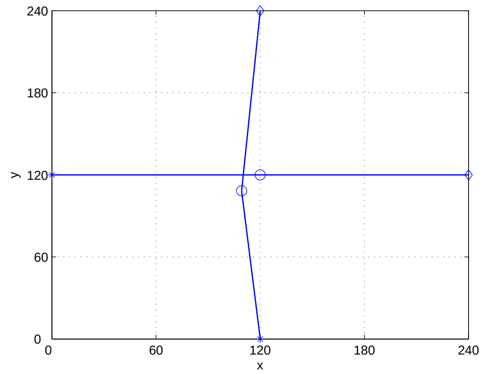

Fig. 3. Optimal resolution manoeuvre calculated with the 2-steps Monte Carlo optimization method.

aircraft 2 must either initially slow down and deviate to the left (way-point in the lower left region) or, symmetrically, accelerate and deviate to the right (way-point in the upper right region). The indetermination due to the symmetrical nature of the problem is naturally solved by the randomized algorithm. The optimal value $u^\star$ of the resolution manoeuvre parameter is estimated as the mode of the sampled density function constructed based on the extractions after the burn-in period. The so-obtained estimate $\hat{u}^\star = (109.3, 108.3)$ corresponds to the resolution manoeuvre depicted in Figure 3. $\hat{u}^\star$ guarantees a performance $perf(\hat{u}^*, \omega) = 0.853$ over the conflict-free joint trajectories $\omega \in \Omega_A$. Also, the fraction of joint trajectories that are conflict-free when adopting the resolution manoeuvre with parameter $\hat{u}^\star$ is larger than that required by the constraint $P_c(\hat{u}^\star) \leq \bar{P} = 0.1$. The conflict probability estimated with accuracy 0.01 and confidence 0.95 by the standard Monte Carlo method is in fact $\hat{P}_c(\hat{u}^*) = 0.59 \cdot 10^{-3}$, so that the fraction of conflict-free trajectories is larger than 98% with probability 0.95.

In the proposed approach to improve the Monte Carlo optimization method, we first compute the optimal deterministic resolution manoeuvre by solving (11) with $\mu_2 = 10^{-6}$ and $\mu_1 = 1 - \mu_2$, so that aircraft 2 takes the responsibility for avoiding conflicts. The so-obtained value $\bar{u}^\star =$

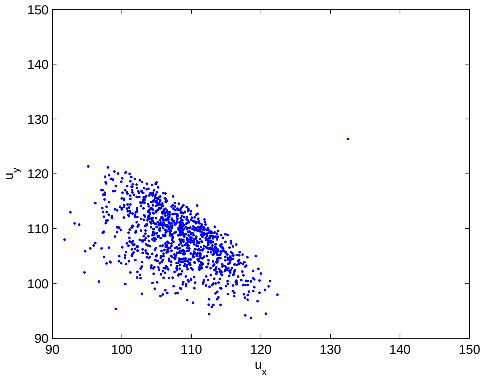

Fig. 4. Values of $u = (u_x, u_y)$ extracted with the proposed method.

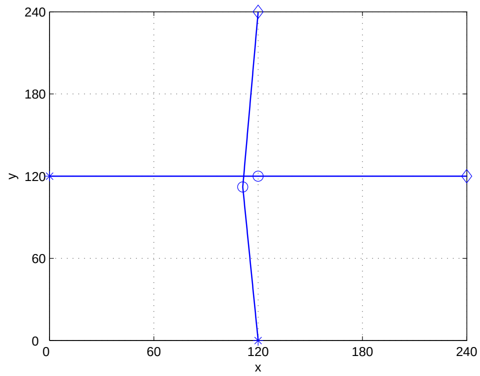

Fig. 5. Optimal resolution manoeuvre calculated with the proposed method.

$(113.4, 113.4)$ is used for setting the instrumental density in the MH algorithm with $M = 20$. The so-obtained parameter $\hat{u}^\star = (111.1, 112.1)$ corresponds to $perf(\hat{u}^\star, \omega) = 0.888$, $\omega \in \Omega_A$, and $\hat{P}_c(\hat{u}^\star) = 0.70 \cdot 10^{-3}$. Figures 4 and 5 represent the values extracted for u and the resolution manoeuvre corresponding to $\hat{u}^\star$, respectively.

The computed $\hat{u}^\star$ differs from $\bar{u}^\star$ obtained with the deterministic method only, due to the uncertainty effecting the actual trajectories of the two aircraft. With respect to the 2-step Monte Carlo optimization method, a significant reduction of the number of necessary simulations is achieved, since simulations are no longer needed to tune the instrumental density, and a comparably lower M ($M = 20$ instead of 100) can be employed in the second phase. Notice that, if one were to use a one-step Monte Carlo optimization with uniform initial density and $M = 20$, a scattered distribution would result as shown in Figure 6, with no evidence of the optimal u.

IV. Conclusions

The Monte Carlo optimization approach to conflict resolution proposed in [3] allows to use accurate and complex stochastic models for the prediction of future aircraft positions. The downside of this approach is its high computational load, mainly related to the huge number of simulations needed

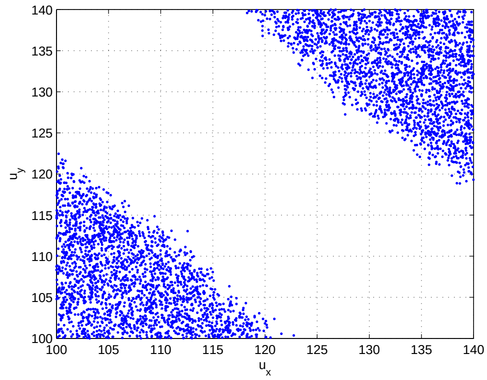

Fig. 6. Values of $u = (u_x, u_y)$ extracted with a 1-step Monte Carlo optimization.

to achieve sufficiently low conflict probability and optimal performance. In this paper, we pointed out that the computational load can be significantly reduced if the Monte Carlo approach is combined with a deterministic approach to conflict resolution that disregards the prediction uncertainty when computing the optimal resolution manoeuvre.

From a different perspective, the combined approach to conflict resolution can be viewed as a way to robustify against uncertainty the design of resolution manoeuvre based on the future nominal aircraft positions.

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